A Comparison of Plausibility Conflict and of Conflict Based on Amount of Uncertainty of Belief Functions

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(A draft of a technical report)

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Abstract. When combining belief functions by conjunctive rules of combination, conflicts often appear, which are assigned to $\emptyset$ by un-normalized conjunctive rule $\circledcirc$ or normalized by Dempster's rule of combination $\oplus$ in Dempster-Shafer theory. Combination of conflicting belief functions and interpretation of conflicts is often questionable in real applications, thus a series of alternative combination rules was suggested and a series of papers on conflicting belief functions was published.
This theoretical contribution presents one of the perspective recent approaches — authors's plausibility conflict — and Harmanec's approach which is, unfortunately, aside the recent interest: conflict based on uncertainty measure and Dempster's rule. Both the approaches are analysed and compared here.
As the approaches are based on completely different assumptions, some of their properties are very different almost counter-intuitive for the first view; on the other hand, the approaches have some analogous properties, which differs both of them from the other commonly used approaches to conflict between belief functions.

Keywords: belief function, Dempster-Shafer theory, uncertainty, internal conflict, conflict between belief functions.

1 INTRODUCTION

Belief functions are one of the widely used formalisms for uncertainty representation and processing that enable representation of incomplete and uncertain knowledge, belief updating, and combination of evidence. They present a principal notion of the Dempster-Shafer Theory or the Theory of Evidence [26].

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When combining belief functions (BFs) by the conjunctive rules of combination, conflicts often appear which are assigned to $\emptyset$ by non-normalized conjunctive rule $\textcircled{\cap}$ or normalized by Dempster's rule of combination $\oplus$. Combination of conflicting BFs and interpretation of conflicts is often questionable in real applications, thus a series of alternative combination rules was suggested and a series of papers on conflicting belief functions was published, e.g. [**?**,**?**,4, 6, 11, 12, 22, 23, 25, 27].

The sum of products of conflicting masses is called *weight of conflict between belief functions* Bel_1 *and* Bel_2 in [26]; this interpretation is commonly used when dealing with conflicting belief functions. Unfortunately, the name and interpretation of this notion does not correctly correspond to reality. We often obtain positive sum of conflicting belief masses even if two numerically same belief functions[1] are combined, see e.g. examples discussed by Almond [1] already in 1995 and by W. Liu [22] in 2006, for another examples see [6].

Liu further correctly demonstrates [22] that neither distance nor difference are adequate measures of conflicts between BFs. Thus she uses a two-dimensional (composed) measure *degree of conflict* $cf(m_1, m_2) = (m_{\textcircled{\cap}}(\emptyset), difBet_{m_1}^{m_2})$ Liu puts together two previous measures of conflict, which are non-adequate separately, $m_{\textcircled{\cap}}(\emptyset)$ and a distance together as two components of a new measure of conflict between BFs cf; unfortunately this does not capture a nature of conflictness / non-conflictness between BFs.

New important and progressive idea comes from athor's [6]. *Internal conflicts* $IntC(m_i)$ which are included in particular individual BFs are distinguished from *conflict between BFs* $C(m_1, m_2)$ in [6]; the entire sum of conflicting masses is called *total conflict* there; and three approaches to conflicts were introduced: combinational, plausibility and comparative. In this study, we will discuss the most elaborated and most prospective of the three approaches — the plausibility conflict, see also **??**.

An *internal conflict* of a BF is a conflict included inside an individual BF. BF is non-conflicting if it is consistent (it has no internal conflict) otherwise it is internally conflicting. A *conflict between BFs* is a conflict between opinions of believers which are expressed by the BFs (the individual attitudes of believers; particular BFs may be internally conflicting or non-conflicting). If there is a positive conflict between BFs, we simply say that the *BFs are mutually conflicting*; otherwise they are *mutually non-conflicting*, i.e., there is no conflict between them.

Analogously to the original $m_{\textcircled{\cap}}(\emptyset)$ and cf, three approaches from [6], including the plausibility conflict (Def. 1 and 2), seem to be rather empirical. For introductive axiomatic studies of conflicts between BFs see [12] and [23], unfortunately these studies do not yet capture a real nature of conflict, as e.g. Martin adds a non-correctly presented or ad-hoc strong axiom of inclusion [23] and proposes an inclusion-weighted distance as a measure of conflict. Hence, this

[1] All BFs combined by $\oplus$ and $\textcircled{\cap}$ are assumed to be mutually independent, even if they are numerically same.
$m(\emptyset)$ is called *autoconflict* when numerically same belief functions are combined [23].

interesting and complex topic is still open for discussion and further development. The important ideas from [12] and [23] should be studied and elaborated together with those from [6].

Unfortunately all the above approaches to conflict of belief functions ignore Harmanec's conflict between BFs which is based on measure of uncertainty and Dempster's rule [17] comming from theory of information. As Harmanec's approach is out of the scope of the above mentioned work on conflicts; and despite the complete different foundation it has some features common with the plausibility conflict. We will analyze it and compare with plausibility conflict here.

2 PRELIMINARIES

2.1 General Primer on Belief Functions

We assume classic definitions of basic notions from theory of *belief functions* [26]. on finite frames of discernment $\Omega_n = \{\omega_1, \omega_2, ..., \omega_n\}$, see also ... .

A *basic belief assignment (bba)* is a mapping $m : \mathcal{P}(\Omega) \longrightarrow [0,1]$ such that $\sum_{A\subseteq\Omega} m(A) = 1$; the values of the bba are called *basic belief masses (bbm)*. $m(\emptyset) = 0$ is usually assumed. A *belief function (BF)* is a mapping $Bel : \mathcal{P}(\Omega) \longrightarrow [0,1]$, $Bel(A) = \sum_{\emptyset\neq X\subseteq A} m(X)$. A *plausibility function* $Pl(A) = \sum_{\emptyset\neq A\cap X} m(X)$. There is a unique correspondence among m and corresponding Bel and Pl thus we often speak about m as of belief function.

A *focal element* is a subset X of the frame of discernment, such that $m(X) > 0$. If all the focal elements are *singletons* (i.e. one-element subsets of Ω), then we speak about a *Bayesian belief function* (BBF); in fact, it is a probability distribution on Ω. In the case of $m(\Omega) = 1$ we speak about *vacuous BF* (VBF).

Dempster's (conjunctive) rule of combination $\oplus$ is given as $(m_1 \oplus m_2)(A) = \sum_{X\cap Y=A} K m_1(X) m_2(Y)$ for $A \neq \emptyset$, where $K = \frac{1}{1-\kappa}$, $\kappa = \sum_{X\cap Y=\emptyset} m_1(X) m_2(Y)$, and $(m_1 \oplus m_2)(\emptyset) = 0$, see [26]; if $\kappa > 0$ then we say that m_1 and m_2 are *combinable* (by Dempster's rule), see [17]. Putting $K = 1$ and $(m_1 \oplus m_2)(\emptyset) = \kappa$ we obtain the *non-normalized conjunctive rule of combination* $\bigcirc\!\!\!\!\cap$, see e. g. [**?**].

Normalized plausibility of singletons[2] of Bel is BBF such that $\frac{Pl(\{\omega_i\})}{\sum_{\omega\in\Omega} Pl(\{\omega\})}$; the formula is also used as definition of probability transformation Pl_P of BF Bel: $(Pl\_P(Bel))(\omega_i) = \frac{Pl(\{\omega_i\})}{\sum_{\omega\in\Omega} Pl(\{\omega\})}$ [2, 5].

2.2 Belief Functions on two-element frame of Discernment

Our analysis of conflicts is motivated by Hájek-Valdés algebraic analysis of BFs on 2-element frame $\Omega_2 = \{\omega_1, \omega_2\}$ [15, 16], further elaborated by the author of this study, e.g. in [3, **?**]. Thus we present some of related notions which are used here.

There are only three possible focal elements $\{\omega_1\}, \{\omega_2\}, \{\omega_1, \omega_2\}$ and any normalized *basic belief assignment (bba)* m is defined by a pair $(a, b) = (m(\{\omega_1\}),$

[2] Plausibility of singletons is called *contour function* by Shafer in [26], thus $Pl\_P(Bel)$ is a normalization of contour function in fact.

$m(\{\omega_2\}))$ as $m(\{\omega_1, \omega_2\}) = 1 - a - b$; this is called *Dempster's pair* or simply *d-pair* in [3, **?**,15, 16] (it is a pair of reals such that $0 \leq a, b \leq 1, a + b \leq 1)^3$.

Extremal d-pairs are the pairs corresponding to BFs for which either $m(\{\omega_1\}) = 1$ or $m(\{\omega_2\}) = 1$, i.e., $(1, 0)$ and $(0, 1)$. The set of all non-extremal d-pairs is denoted as D_0; the set of all non-extremal *Bayesian d-pairs* (i.e. d-pairs corresponding to Bayesian BFs, where $a + b = 1$) is denoted as G; the set of d-pairs such that $a = b$ is denoted as S, the set where $b = 0$ as S_1, analogically the set where $a = 0$ as S_2 (simple support BFs). Vacuous BF is denoted as $0 = (0, 0)$ and there is a special BF (d-pair) $0' = (\frac{1}{2}, \frac{1}{2}) = U_2$, see Figure 1. (VBF 0 is neutral w.r.t. Dempster's rule, i.e. for any BF Bel it holds that $Bel \oplus 0 = Bel = 0 \oplus Bel$; similarly $0'$ is neutral in G, i.e., $(a, 1-a) \oplus 0' = (a, 1-a) = 0' \oplus (a, 1-a)$, and generally $Bel \oplus U_n = Bel = U_n \oplus Bel$ for any BBF Bel on Ω_n).

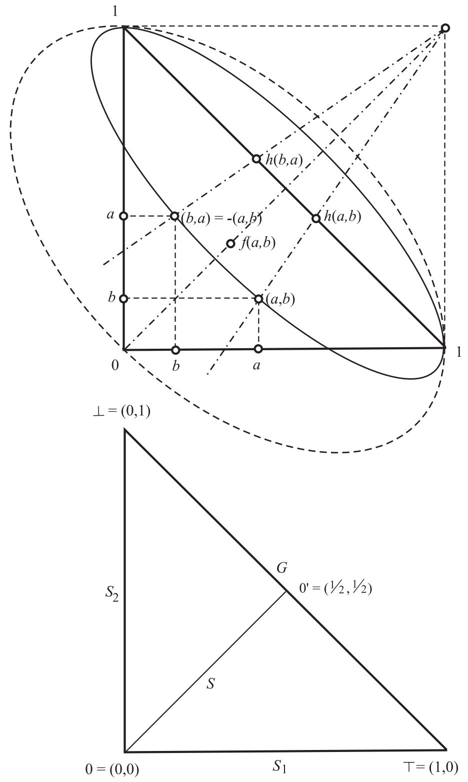

Fig. 1. Dempster's semigroup D_0. Homomorphism h is in this representation a projection of D_0 to group G along the straight lines running through the point $(1, 1)$.

[3] Analogically, we can represent any BF on Ω_n as a 2^{n-2}-tuple $(a_1, a_2, ..., a_{2^n-2})$, or as a 2^{n-1}-tuple $(a_1, a_2, ..., a_{2^n-2}; a_{2^n-1})$ if we want to underline value $m(\Omega) = a_{2^n-1}$. For non-normalized BFs we can use $(a_1, a_2, ..., a_{2^n-2}; a_{2^n-1} \,|\, e)$, where $e = m(\emptyset)$.

In D_0, we need further: $h(a,b) = (a,b) \oplus 0' = (\frac{1-b}{2-a-b}, \frac{1-a}{2-a-b})$, in general $h(Bel) = Bel \oplus U_n = Pl\_P(Bel)$. h is an homomorphism of the algebraic structure on D_0 to G.
Let us denote $D_0^{\geq 0} = \{(a,b) \in D_0 \mid (a,b) \geq 0$, i.e.,$a \geq b\}$ and analogically $D_0^{\leq 0'} = \{(a,b) \in D_0 \mid (a,b) \leq 0'$, i.e., $a \leq b\}$. And analogically subsets of G: $G^{\leq 0'}$ and $G^{\geq 0'}$; $G^{\leq 0'} = \{(a, 1-a) \in D_0 \mid (a, 1-a) \leq 0'$, i.e., $a \leq 0.5\}$, $G^{\geq 0'} = \{(a, 1-a) \in D_0 \mid (a, 1-a) \geq 0'$, i.e.,$a \geq 0.5\}$.

For more details and algebraic results see [3, **?**,15, 16] For the first results of generalization to Ω_3 see [8].
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The situation is much more complicated there, as instead of 2-dimensional triangle for Ω_2 there is 6-dimensional simplex for Ω_3, there are two kind of dimensions, and adequately more complicated structures.

3 PLAUSIBILITY CONFLICT OF BELIEF FUNCTIONS

Two BFs on a two-element frame of discernment which both support/prefer the same element of the frame ($m_i(\omega_j\} \geq \frac{1}{2}$ for same $\omega_j \in \Omega_2$), i.e., both oppose the other element ($m_i(\omega_k\} \leq \frac{1}{2}$ for $\omega_j \neq \omega_j \in \Omega_2$), are assumed to be mutually non-conflicting in [6] (there is no conflict between them); otherwise they are mutually conflicting. A generalization of this idea follows.

3.1 Internal Plausibility Conflict

Definition 1. *The* internal plausibility conflict *Pl-IntC* of BF Bel *on a general frame of discernment Ω is defined as*

$$Pl\text{-}IntC(Bel) = 1 - max_{\omega \in \Omega} Pl(\{\omega\}),$$

where Pl is the plausibility corresponding to Bel.

Let us present the plausibility internal conflict on n-element frame of discernment Ω_n. $0'_n = (\frac{1}{n}, \frac{1}{2}, ..., \frac{1}{n}, 0, 0, ..., 0) = U_n$ has maximal internal conflict: $Pl\text{-}IntC(U_n) = \frac{n-1}{n}$, whereas categorical BFs, simple support BFs, consosant and any consistent BFs have no (i.e., zero) internal conflict $Pl\text{-}IntC$.

Situation of a special case of plausibility internal conflict of BFs on Ω_2 is graphically presented in Figure 2. The directions of the arrows show the directions in which internal conflict decreases. A lines without arrows along S_1 and S_2 represent constant (zero) internal conflict of BFs from these subsemigroups, dashed lines represent positive constant internal conflict.

3.2 Plausibility Conflict between Belief Functions

Definition 2. *The* conflicting set $\Omega_{PlC}(Bel_1, Bel_2)$ *is defined as the set of* elements $\omega \in \Omega_n$ with conflicting Pl_P masses*, i.e.,* $\Omega_{PlC}(Bel_1, Bel_2) = \{\omega \in$

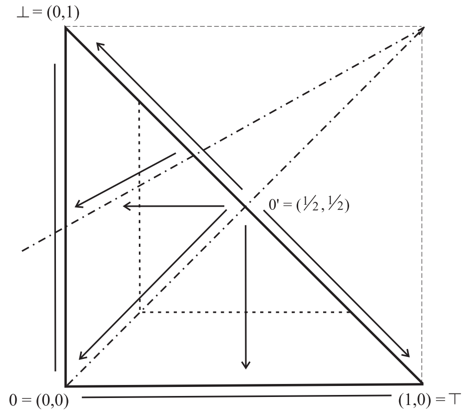

Fig. 2. Plausibility internal conflict

$\Omega_n \mid (Pl\_P(Bel_1)(\omega) - \frac{1}{n})(Pl\_P(Bel_2)(\omega) - \frac{1}{n}) < 0\}$.
Plausibility conflict between BFs Bel_1 *and* Bel_2 *is then defined by the formula*

$$Pl\text{-}C(Bel_1, Bel_2) = min(\, Pl\text{-}C_0(Bel_1, Bel_2), (m_1 \odot m_2)(\emptyset)\,),$$

where[4]

$$Pl\text{-}C_0(Bel_1, Bel_2) = \sum_{\omega \in \Omega_{PlC}(Bel_1, Bel_2)} \frac{1}{2} |Pl\_P(Bel_1)(\omega) - Pl\_P(Bel_2)(\omega)|.$$

If $(Pl\_P(Bel_1)(\omega_i) - \frac{1}{n})(Pl\_P(Bel_2)(\omega_i) - \frac{1}{n}) \geq 0$ for all $\omega_i \in \Omega_n$, i.e., $\Omega_{PlC}(Bel_1, Bel_2) = \emptyset$ [5], then BFs Bel_1 and Bel_2 on Ω_n are mutually non-conflicting. The reverse statement does not hold true for $n > 2$, see e.g. Example 1. Any two BFs $(m_1(\{\omega_1\}), m_1(\{\omega_2\})) = (a, b)$ and $(m_2(\{\omega_1\}), m_2(\{\omega_2\})) = (c, d)$ on Ω_2 are mutually non-conflicting iff $(a - b)(c - d) \geq 0$.

Contrary to the use of $m_{\odot}(\emptyset)$, degree of conflict cf or measures of conflict based on a distance, when using the plausibility conflict, two BFs which accordingly support/oppose same elements of a frame of discernment with a different degree of support/opposition are not misclassified as being in mutual conflict.

Example 1. Let us suppose Ω_6, now; and two intuitively non-conflicting BFs m_1 and m_2.

X :	$\{\omega_1\}$	...	$\{\omega_1, \omega_2, \omega_3, \omega_4\}$
$m_1(X)$:	1.00		
$m_2(X)$:			1.00

$Pl\_P(m_1) = (1.00, 0.00, 0.00, 0.00, 0.00, 0.00)$,

[4] $Pl\text{-}C_0$ is not a separate measure of conflict in general; it is just a component of $Pl\text{-}C$.
[5] For improvement of a construction of $\Omega_{PlC}(Bel_1, Bel_2)$ for more complicated situations see [10];

$Pl\_P(m_2) = (0.25, 0.25, 0.25, 0.25, 0.00, 0.00)$, (we mean $Pl\_P(Bel_i)$ for Bel_i corresponding to m_i), $\Omega_{PlC}(m_i, m_j) = \{\omega_2, \omega_3, \omega_4\}$, as $Pl\_P(m_2)(\omega_i) = \frac{1}{4} > \frac{1}{6}$ for $i = 2, 3, 4$, whereas $Pl\_P(m_1)(\omega_i) = 0 < \frac{1}{6}$ for $i = 2, 3, 4$, (the other elements are non-conflicting: $Pl\_P(m_1)(\omega_1) = 1 > \frac{1}{6}$, $Pl\_P(m_2)(\omega_1) = \frac{1}{4} > \frac{1}{6}$, $Pl\_P(m_1)(\omega_i) = 0 = Pl\_P(m_2)(\omega_i)$ for $i = 5, 6$; $Pl\text{-}C(m_1, m_2) = min(0.375, 0.00) = 0.00$.

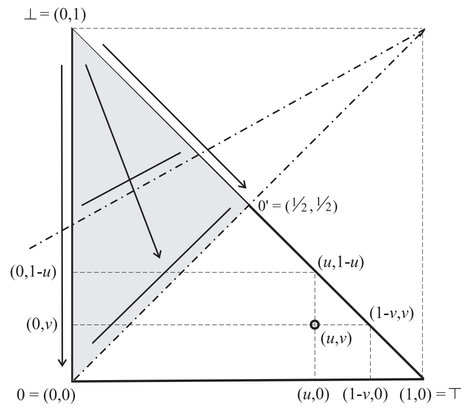

Fig. 3. Plausibility conflict between fixed BF (u, v) and general BF (a, b) on Ω_2; $Pl\text{-}C$ decreases in direction of arrows and it is constant along lines without arrows.

Plausibility conflict between fixed (u, v) on Ω_2 and free (a, b) is presented on Figure 3. There is no plausibility conflict between (u, v) and any BF (a, b) such that $(u - v)(a - b) \geq 0$, i.e., when (a, b) is in the same subsemigroup $D_0^{\geq 0}$ or $D_0^{\leq 0'}$ as (u, v) is, (see the white area on Figure 3). On the other hand, there is positive plausibility conflict between (u, v) and any BF (a, b) such that $(u - v)(a - c) < 0$ (see the grey area). $Pl\_P(u, v) = h(u, v) = (\frac{1-v}{2-u-v}, \frac{1-u}{2-u-v})$, similarly for (a, b), BFs are plausibility non-conflicting if and only if $(\frac{1}{2} - h_1(u, v))(\frac{1}{2} - h_1(a, b)) \geq 0$, thus iff $(u - v)(a - b) \geq 0$.

Plausibility conflict between (u, v) and (a, b) increases from $|\frac{1}{2} - \frac{1-u}{2-u-v}|$ to $|\frac{1}{2} - \frac{1-u}{2-u-v}| + \frac{1}{2}$ for any BFs from G, S_i; in detail from ϵ surrounding of $0'$ to the corresponding conflicting extremal BF in G, respectively from ϵ surrounding of 0 to the corresponding conflicting extremal BFs in $S_i's$. Similarly, $Pl\text{-}C((u, v), (a, b))$ increases for BFs on h-lines closer to the corresponding conflicting extremal element, while conflict between (u, v) and (a, b) is same for all BFs laying on the same h-line, see Figure 3, arrows represent decreasing of conflicts between (a, b) and (u, v), in the grey area ($D_0^{\leq 0}$) which contains BFs conflicting with given (u, v).

Plausibility conflict between general BFs Bel and a given Bel_{UV} on Ω_n increases from $Pl\text{-}C(Bel_{UV}, U_n)$ to $Pl\text{-}C(Bel_{UV}, U_n) + \frac{n-1}{n}$ for any BFs from ϵ surroundings of 0, U_n and indecisive BFs to the corresponding conflicting categorical BF. $Pl\text{-}C(Bel, Bel_{UV})$ is constant for all BFs with the same $h(Bel)$.

4 CONFLICT BETWEEN BELIEF FUNCTIONS BASED ON UNCERTAINTY AND THE DEMPSTER RULE

In this section we will recall the measure of uncertainty for Dempster-Shafer theory justified by Harmanec and Klir in [18], for its efficient algorithm see [19], relation of this measure to the Dempster rule and the measure of the conflict between belief functions based on the uncertainty and the Dempster rule [17].

4.1 A Relation of Uncertainty and the Dempster Rule

Definition 3. *Let Bel denote a belief function defined on a general frame of discernment Ω. A measure of the amount of uncertainty contained in Bel, denoted as $AU(Bel)$, is defined by*

$$AU(Bel) = max \left\{ - \sum_{\omega \in \Omega} p_\omega \; log_2 \; p_\omega \right\},$$

where the maximum is taken over all $\{p_\omega\}_{\omega\in\Omega}$ such that $p_\omega \in [0,1]$ for all $\omega \in \Omega$, $\sum_{\omega\in\Omega} p_\omega = 1$, and for all $A \subseteq \Omega$, $Bel(A) \leq \sum_{\omega\in A} p_\omega$.

For comparison of both the presented approaches to conflict, the following necessary and sufficient condition for no increase of uncertainty after Dempster's combination on a two-element frame of discernment is useful.

Theorem 1. *Let us suppose two combinable belief functions Bel_1 and Bel_2 on a two-element frame of discernment $\Omega_2 = \{\omega_1, \omega_2\}$, given by d-pairs (a_1, b_1) and (a_2, b_2); assume further $a_1 \geq b_1$, i.e., $(a_1, b_2) \geq 0$. Then*

$$AU(Bel_1 \oplus Bel_2) \leq min(AU(Bel_1), AU(Bel_2))$$

if and only if at least one of the following holds
(i) $0 \leq a_1, a_2, b_2 \leq \frac{1}{2}$, *(i.e. also* $0 \leq b_1 \leq \frac{1}{2}$*); (see Fig. 6)*
(ii) $a_2 \geq b_2$; *(see Fig. 5)*
(iii) $a_2 < b_2$, $(1-b_1)(1-b_2) \geq (1-a_1)(1-a_2)$,
$a_2(1-b_1) \geq a_1 b_2$,
$(1-b_2)(1-a_1b_2-b_1a_2) \geq (1-a_1)(1-a_2)$; *or*
(iv) $a_2 < b_2$, $(1-b_1)(1-b_2) < (1-a_1)(1-a_2)$,
$b_1(1-a_2) \geq a_1 b_2$,
$(1-a_1)(1-a_1b_2-b_1a_2) \geq (1-b_1)(1-b_2)$.

For proof see [17]. An analogous but more complicated necessary and sufficient conditions for three-element frames of discernment are mentioned, but not presented there.

4.2 Harmanec's Conflict between Belief Functions

Hamanec's definition of conflict between belief functions is motivated by the above presented quite complex (and not easy to grasp) relation between uncertainty and the Dempster's rule. Harmanec uses the increase of uncertainty as the defining property of conflict between bodies of evidence.

Definition 4. *Let Bel_1 and Bel_2 denote combinable belief functions on Ω.*

We define the degree of conflict of Bel_1 and Bel_2 denoted $\mathbb{C}(Bel_1, Bel_2)$, by

$$\mathbb{C}(Bel_1, Bel_2) =$$
$$max\,(0, AU(Bel_1 \oplus Bel_2) - min_i AU(Bel_i))\,.$$

That is the degree of conflict is equal to the amount of uncertainty gained (or, equivalently, the amount of information lost[6]) by Dempster's combination $Bel_1 \oplus Bel_2$ (a conflict $\mathbb{C}(Bel_1, Bel_2, ..., Bel_n)$ of n belief functions is defined analogously).

5 A COMPARISON OF THE APROACHES

5.1 Uncertainty and Internal Conflict

Unlike authorl's approach, there is no internal conflict specified in Harmanec's approach. On the other hand, there is uncertainty of individual beliefs Bel_1 and Bel_2, uncertainty of their combination $Bel_1 \oplus Bel_2$, and conflict between Bel_1 and Bel_2. Thus there is some kind of analogy of both the approaches: where $AU(Bel_i)$ is analogous to internal conflict of Bel_i, i.e. to $IntC(Bel_i)$ and $AU(Bel_1 \oplus Bel_2)$ is analogous to total conflict of Bel_1 and Bel_2, which is equal to $totC(Bel_1, Bel_2) = (m_1 \circledcirc m_2)(\emptyset)$. Moreover Harmanec's conflict is computed using AU, thus this seem useful to compare $AU(Bel)$ with $Pl\text{-}IntC(Bel)$.

We can demonstrate the two-element case of $AU(Bel)$ on Figure 4. For Bayseian BFs it is really analogous to $Pl\text{-}IntC(Bel)$, AU is maximal for $0' = U_2$ ($AU(U_2) = 1 = log_2\; n = log_2\; 2$) and it decreases to 0 towards both $(0, 1)$ and $(1, 0)$. On the other hand $AU(Bel)$ is completely different for $Bel = (s, s) \in S$, it is not decreasing towards VBF $0 = (0, 0)$, but constant $AU(Bel) = 1$ for all $Bel \in S$. VBF is completely without any internal conflict, but it has maximal uncertainty $AU(VBF) = 1$. Non-analogous are also all simple (support) BFs $(a, 0) \in S_1$ and $(0, b) \in S_2$, they are decreasing to 0 with increasing a (with increasing b), but $Pl\text{-}IntC(a, 0) = Pl\text{-}IntC(0, b)$ are constantly equal to 0. Subsequently for $a > \frac{1}{2}$, $a > b$, $AU(a, b)$ decreases towards right along horizontal lines parallel with S_1 and they are constant on vertical lines parallel with S_2, what is conversely for $Pl\text{-}IntC(a, b)$. Analogously, but conversely for $a < b$, $b > \frac{1}{2}$. Big difference is also maximal uncertainty $AU(a, b) = 1$ for all BFs such that $0 \leq a, b \leq \frac{1}{2}$.

[6] The information gain $\mathcal{G}(Bel_1, Bel_2)$ is defined dually in (Harmanec, 1997).

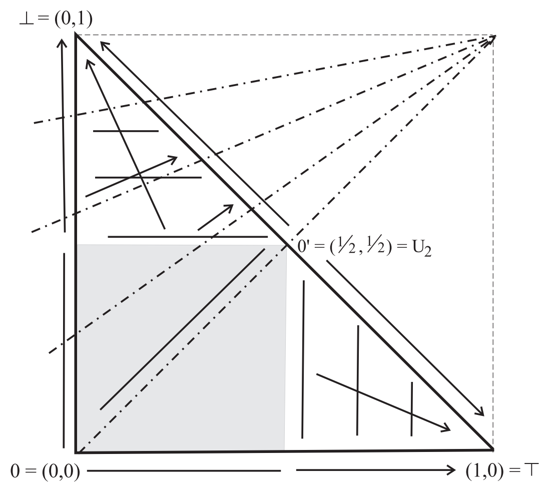

Fig. 4. Uncertainty $AU(Bel)$ of $Bel = (a, b)$ on Ω_2.; uncertainty decreases in direction of arrows; it is constant along the lines without arrows.

Let us turn our attention to a general n-element case (BFs on a n-element frame of discernment Ω_n) now. In the case of Bayesian BFs, AU is maximal for U_n ($AU(U_n) = log_2\ n$) and it decreases towards categorical Bayesian BFs ($m(\omega_i) = 1$ for some $\omega_i \in \Omega_n$). For general BFs, AU is maximal for all symmetric BFs, for all qBBFs such that $m(\omega_i) \leq \frac{1}{n}$, and for some other BFs (it is not easy to explicitly enumerate all of these BFs with uncertainty equal to $log_2\ n$); and AU decreases towards categorical BFs with singleton focal element. Note that $AU(Bel_{C2}) = 1$, for a categorical BF with two-element focal element, for any frame of discernment.

Thus the analogy of $AU(Bel)$ and $Pl\text{-}IntC(Bel)$ is very weak in general.

5.2 Analysis of Conflict between BFs on Ω_2

Let us turn our attention to conflict between belief functions now. We will start with mutual conflictness / non-conflictness of two BFs on two-element frame Ω_2.

This question is very easy in the case of plausibility conflict $Pl\text{-}C$. Two BFs $Bel_i = (a_i, b_i)$ on Ω_2 are mutually non-conflicting, i.e., there is no conflict between them if and only if, both of them support same ω_i and both of them oppose the other element of Ω_2 thus if and only if $a_i \geq b_i$ for $i = 1, 2$ or $a_i \leq b_i$ for $i = 1, 2$. I.e., if both of Bel_i are in grey part of the triangle on Figure 5 or both of them are in white part of the triangle. The BFs are mutually conflicting (there is some positive conflict between them) if one of the BFs is in white part and the other in grey part, i.e., if and only if $a_1 > b_1$ & $a_2 < b_2$ or $a_1 < b_1$ & $a_2 > b_2$.

In the case of Harmanec's conflict, we see (from Definition 4) that Bel_i are mutually non-conflicting, i.e., there is no (zero) Harmanec's conflict between Bel_1

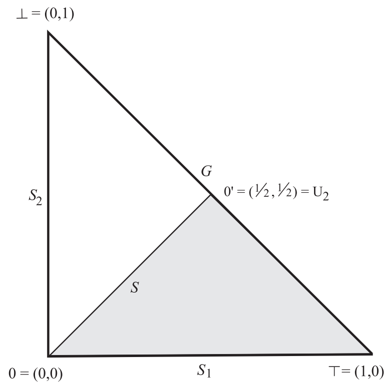

Fig. 5. Belief functions on Ω_2: $a \geq b$, $Bel(\{\omega_1\}) \geq Bel(\{\omega_2\})$.

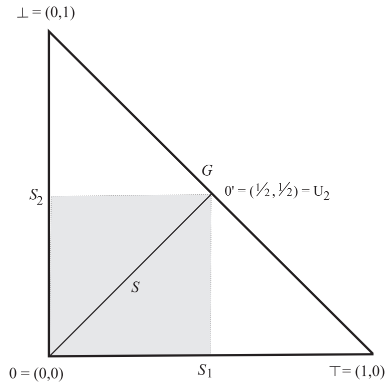

Fig. 6. Belief functions on Ω_2: $0 \leq a, b \leq \frac{1}{2}$, $0 \leq Bel(\{\omega_1\}), Bel(\{\omega_2\}) \leq \frac{1}{2}$.

and Bel_2 if and only if $AU(Bel_1 \oplus Bel_2) \leq AU(Bel_1), AU(Bel_2)$, i.e., if and only if the condition from Theorem 1 is satisfied or if and only if dual condition is satisfied in the case that $a_1 \leq b_1$ holds true. Subcondition (i) says that both BFs are in a/the grey square on Figure 6, subcondition (ii) says that both BFs are in the grey triangle on Figure 5, while its dual subcondition says that both BFs are in the white triangle on the figure (when $a_1 \leq b_1$ holds true). Subconditions (iii) and (iv) are more complicated, to be simply displayed on figures, each of both subconditions is again composed from several simpler conditions in fact. Even from this partial analysis of Harmanec's condition, we can see that for any pair of BFs with zero plausibility conflict there is no (zero) Harmanec's conflict. Thus we have proven the following theorem.

Theorem 2. *Let us suppose two combinable belief functions Bel_1 and Bel_2 on a two-element frame of discernment $\Omega_2 = \{\omega_1, \omega_2\}$, given by d-pairs (a_1, b_1) and (a_2, b_2). If Pl-C$(Bel_1, Bel_2) = 0$ then also $\mathbb{C}(Bel_1, Bel_2) = 0$.*

6.2.1 Conflict between a free (a, b) and a fixed Bayesian $(u, 1-u)$

Look at an analysis of Harmanec's conflict $\mathbb{C}((a, b), (u, v))$ analogous to the analysis of *Pl-C*$((a, b), (u, v))$ in Section 3. As the formula (and procedure) for computation of $\mathbb{C}((a, b), (u, v))$ is significantly more complicated, we will start with a simplified but important case of Bayesian (u, v). Thus we are interesting in conflict between (a, b) and $(u, 1-u)$ for fixed $(u, 1-u)$.

For a special case of Bayesian BF $0' = U_2 = (\frac{1}{2}, \frac{1}{2})$ we have the following lemma.

Lemma 1. *$U_2 = (\frac{1}{2}, \frac{1}{2})$ is non-conflicting with any belief functions on two-element frame of discernment, i.e.. for any Bel on Ω_2 it holds that $\mathbb{C}(Bel, U_2) = 0$.*

Proof. Proof is a verification of the statement, based on the fact that $Bel \oplus U_2 = Pl\_P(Bel)$ Thus $(a,b) \oplus U_2$ lies between $(a, 1-a)$ and $(b, 1-b)$ and has less uncertainty than (a,b) has. $\square$.

Let us suppose $u > \frac{1}{2}$, $v = 1-u$, now, see Figure 7. If $a \geq b$ then $AU((a,b) \oplus (u, 1-u) \leq AU(a,b), AU(u, 1-u)$ according to subcondition (i) from Theorem 1, hence $\mathbb{C}((a,b),(u,1-u)) = 0$. The subcondition (ii) is not relevant as $u > \frac{1}{2}$ now. Maximal uncertainty $AU(\oplus) = 1$ (read: $AU((a,b) \oplus (u, 1-u) = 1)$ appears for $(a,b) = (1-u, u)$ and for all BFs lying on the same h-line as $(1-u, u)$, i.e. such that $Pl\_P(a,b) = (1-u, u)$.

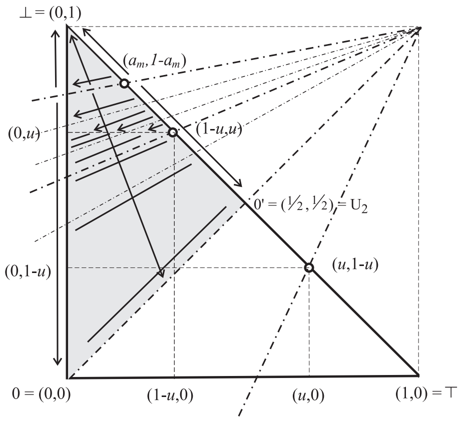

Fig. 7. Harmanec's conflict $\mathbb{C}((a,b), (u, 1-u))$ for $u > 0.618$.

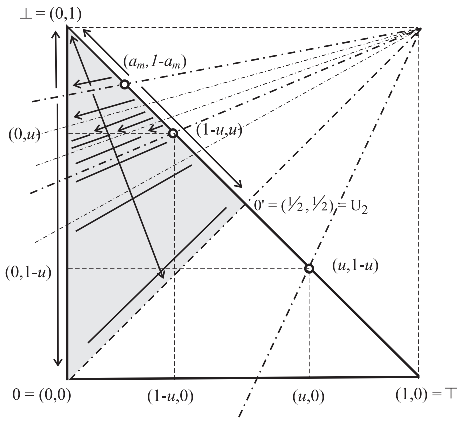

Fig. 8. *** modified Figure for $u \doteq 0.6$ in preparation ***

$\mathbb{C}((a, 1-a),(u, 1-u))$:
Let assume $b = 1-a$ for a moment: $AU(\oplus)$ increases for a decreasing from $\frac{1}{2}$ to $1-u$, $AU(a, 1-a) \geq AU(u, 1-u)$ there (for $a \in [1-u, \frac{1}{2}]$), thus conflict $\mathbb{C}((a, 1-a),(u, 1-u)) = AU(\oplus) - AU(u, 1-u)$ increases with uncertainty from 0 for a decreasing from $\frac{1}{2}$ to $1-u$. For $a \leq 1-u$, $\mathbb{C}((a, 1-a),(u, 1-u)) = AU(\oplus) - AU(a, 1-a)$, both $AU(\oplus)$ and $AU(a, 1-a)$ decrease there, $AU(a, 1-a)$ decreases more when closer to $(1-u, u)$ thus the conflict still increases till its maximum for $(a_m, 1-a_m)$, $0 < a_m < 1-u$. Further it decreases till zero for $a = 0$. We can show that the conflict is positive for any $a > 0$: $(a, 1-a) \oplus (u, 1-u) = (\frac{au}{1-a-u+2au}, \frac{(1-a)(1-u)}{1-a-u+2au})$, $\frac{au}{1-a-u+2au} \leq a$ (eq. for $a = 0$), thus $AU(\oplus) \leq AU(a, 1-a)$, where equality holds for $a = 0$ in our range ($a \leq 1-u$); thus conflict is positive for $a > 0$. This is represented by arrows from $(a_m, 1-a_m)$ to $(0,1)$ and to U_2 in Figure 7.

$\mathbb{C}((a,b),(u, 1-u))$ for $a < b$, $b \leq u$:
In this case $AU(u, 1-u) \leq AU(a,b)$, thus $\mathbb{C}((a,b),(u, 1-u)) = AU(\oplus) - AU(u, 1-u)$ now. We subtract fixed uncertainty (of fixed $(u, 1-u)$), thus conflict is constant for all BFs on the same h-line analogously to the case of h-line

containing $(a_m, 1-a_m)$. This is represented by lines without arrows in directions of h-lines. The conflict is decreasing for h-lines closer to S, this is illustrated by an arrow intersecting h-lines without arrows.

$\mathbb{C}((a,b),(u,1-u))$ for $a < b$, $b \geq u$:
In this case $AU(a,b) \leq AU(u,1-u)$, thus $\mathbb{C}((a,b),(u,1-u)) = AU(\oplus) - AU(a,b)$ now. This case is analogous to the previous, but the uncertainty which is decreased is not constant, it is increasing with with decreasing b. Thus neither conflict on a h-line is not constant but decreasing with decreasing b. This is represented by arrow in directions of h-lines. The conflict is further decreasing from h-line containing $(a_m, 1-a_m)$ toward $(0,1)$, this is represented by an arrow there.

$\mathbb{C}((0,b),(u,1-u))$:
For $b \leq u$ the situation is easy, fully described above. The conflict $\mathbb{C}((0,b),(u,1-u))$ is the same as on the h-line intersecting the triangle in $(0,b)$. Thus it decreases from the intersection of the triangle with h-line containing $(a_m, 1-a_m)$ both toward $0 = (0,0)$ and toward $(0,1)$, see arrows on the figure.
The situation is more complicated for $b \geq u$, the conflict is decreasing along h-lines. Can the conflict decrease to zero? The answer depends from the specific value of u. It is possible to show that for $u > \frac{1}{2}(\sqrt{5}-1) \doteq 0.618034$ the conflict is always positive. On the other side it can decrease to zero for $u < 0.618$; the subcondition (iv) from Theorem 1 is satisfied there[7]. We need a modified figure, see Figure 8, containing this non-conflicting area, for $\frac{1}{2} < u < 0.614$.

6.2.2 Conflict between a free (a,b) and a fixed general (u,v)

Let us start with a special case again. The following generalization of Lemma 1 holds true:

Lemma 2. *Any symmetric belief function $Sym = (s,s)$ is non-conflicting with any other belief function on two-element frame of discernment, i.e.. for any Bel and any symmetric BF Sym both on Ω_2 it holds that $\mathbb{C}(Bel, Sym) = 0$.*

Proof. We can make a more complicated analogy of the previous proof; or less ellegantly but simply apply subconditions (i) and (ii) from Theorem 1 (this is also an alternative proof of Lemma 1). $\square$.

Let us assume that $u > v$, see Figure 9 now. We can proceed analogously to the previous case of $(u,1-u)$. But the situation is more complicated. In Dempster's combination $Pl\_P(u,v)$ plays principal role, whereas at $AU(u,v)$ directly u plays principal role, hence there is more important points in the figure. And the behaviour of conflict is correspondingly more complicated.

[7] Satisfying subconditions from from Theorem 1: subcondition (i) is satisfied for $(a \geq b)$, subcondition (ii) can be satisfied only in the special case $(u,1-u) = U_2$ in the case of fixed Bayesian $(u,1-u)$, subcondition (iv) can be satisfied only in case $u \leq 0.618$, subcondition (iii) is not satisfied or it is covered by the other subconditions.

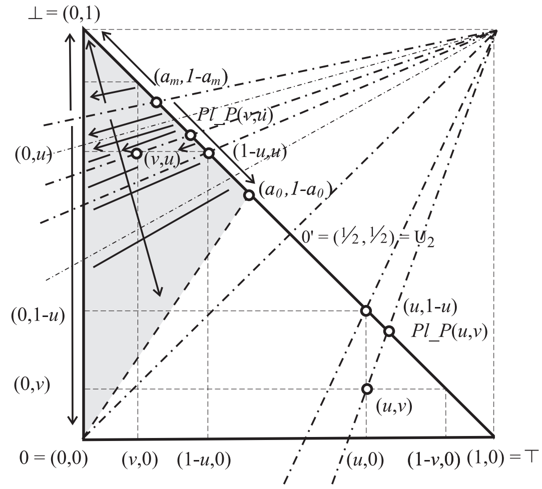

Fig. 9. Harmanec's conflict $\mathbb{C}((a,b),(u,v))$.

$\mathbb{C}((a,1-a),(u,v))$:
Analogously to the previous case we have maximal $AU(\oplus)$ (read as $AU((a,1-a)\oplus(u,v))$ now) for (v,u) and $Pl_P(v,u)$. And maximal conflict for $(a_m,1-a_m$, such that $0<a_m<\frac{1-u}{2-u-v}$, where $Pl\_P(u,v)=(\frac{1-v}{2-u-v},\frac{1-u}{2-u-v})$. The difference from the previous case is close to U_2: $AU(\oplus)$ fall between $(u,1-u)$ and $Pl\_P(u,v)$ for $\frac{u(1-u)}{1-v+uv-u^2}\leq a\leq\frac{1}{2}$, thus there is no conflict for $(a_0,1-a_0)=(\frac{u(1-u)}{1-v+uv-u^2},\frac{(1-u)(1-v)}{1-v+uv-u^2})$ and all Bayesian $(a,1-a)$ between $(a_0,1-a_0)$ and U_2. Sequently there is no conflict either for BFs in corresponding surrounding of S. The subcondition (iii) from Theorem 1 is satisfied here.

$\mathbb{C}((a,b),(u,v))$:
The rest is analogous to the case of $(u,1-u)$.

$\mathbb{C}((0,b),(u,v))$:
In this case, we can show that $\mathbb{C}((0,b),(u,v))$ is positive even in closer neighbourhood of $(0,0)$, that the area of satisfaction of the subcondition (iii) narrows to a single point containing vacuous BF $0=(0,0)$. An analogously to the previous case, that $(0,b)$ is always conflicting with (u,v) for greater u, see Figure 9, i.e. subcondition (iv) cannot be satisfied. And that there appears non-conflicting BFs around $(0,b)$ closer to $(0,1)$ for (u,v) closer to U_2 and to non-conflicting area corresponding to subcondition (i). To capture this we need a modified figure again. (An alternative figure is under development).

We have already seen, that any two *Pl-C* non-conflicting belief functions on Ω_2 are also $\mathbb{C}$ non-conflicting. After the detail analysis of $\mathbb{C}$ this seems very obvious. Using the previous analysis we can show that a stronger statement holds true:

Theorem 3. *Let us suppose two combinable belief functions Bel_1 and Bel_2 given by d-pairs (a_1, b_1) and (a_2, b_2) on a two-element frame of discernment $\Omega_2 = \{\omega_1, \omega_2\}$. It holds that*

$$\mathbb{C}((a_1, b_1), (a_2, b_2)) \leq Pl\text{-}C((a_1, b_1), (a_2, b_2)).$$

Proof. The proof is just a verification of the statement for a pairs of Bayesian BFs. *Pl-C* is the same for any couple on a same couple of h-lines. Whereas $\mathbb{C}$ is either same or less leaving the Bayesian BFs, thus the property is kept or strenghtened leaving Bayesian BFs. $\Box$.

5.3 A Discusion of Harmanec's Conflict and a Comparison of the Approaches on a General Finite Frame of Discernment

Let us outline an analysis of the Harmanec's conflict between BFs on general finite Ω_n in this subsection. Let us start with Bayesian BFs again. $U_n = (\frac{1}{n}, \frac{1}{n}, ..., \frac{1}{n})$ is due to its neutrality w.r.t. to Demspter's combination is non-conflicting with any of the other Bayesian again. Let us start with a fixed Bel_u and a variable Bel_a Bayesian BFs again; analogously to d-pairs we can represent them by n-tuples of m-values of their singletons $m(\{\omega_i\})$:$\mathrm{Bel}_u = (u_1, u_2, ..., u_n)$, where $\sum_i u_i = 1$. If $Bel_a = (a_1, a_2, ..., a_n)$ have the same order of focal element with respect to size of its m-values the uncertainty decreases during $\oplus$ combination, thus BFs are non-conflicting. Similar situation appears when max m-value is assigned to same singletons. On the other side the maximal uncertainty of $AU(\oplus)$ (read $AU(Bel_a \oplus Bel_u)$ now) is obtained for unique $(a_1, a_2, ..., a_n) = -(u_1, u_2, ..., u_n)$, such that $(a_1, a_2, ..., a_n) \oplus -(u_1, u_2, ..., u_n) = U_n$; for uniqueness of this value see [7]. Analogously to Ω_2 conflict increases from U_n to $-(u_1, u_2, ..., u_n)$ and further to belief function(s) Bel_m which is (are) behind $-(u_1, u_2, ..., u_n)$. Conflict further decreases to BBFs with less focal elements, finally to 0 for all categorical BBFs (some of a_i is equal to 1).

A situation starts to be complicated when Bel_a leaves BBFs (when it has also non-singleton focal elements), conflict is constant on h-lines (set of BFs with same $Pl\_P(Bel)$) or decreases with increase of $AU(Bel_a)$ depending from comparison of $AU(Bel_a)$ and $AU(Bel_u)$ in the previous case of Ω_2. On Ω_n, we have many-dimensional structure of $\{Bel \mid Bel \oplus U_n = Pl\_P(Bel_a) = Bel_a \oplus U_n\}$ instead of one-dimensional h-lines. For an introduction on algebra of belief functions on Ω_3 see [8]. The problem is that AU is not always increasing leaving Bayesian BFs in these structures thus conflict does not need to decrease there. Thus we need to distinguish between qBBFs and general BFs, as for qBBFs there are one-dimensional h-lines analogous to the case of Ω_2; h-lines are straight lines going through the point $(1, 1, ..., 1)$ of n-dimensional space now.

6.3.1 Harmanec's Conflict and its Comparison to Plausibility Conflict on Quasi-Bayesian Belief Functions on Ω_n.

Analogously to the simplest case of Ω_2, where are only qBBFs, we can use h-lines defined by homomorpism h again. Similarly to Ω_2, AU decreases along

h-lines in direction to BBFs as less and less iso-AU levels are crossed in this direction, analogously to Figure **??**. On the other side Harmanec's conflict $\mathbb{C}$ is again constant along h-lines or it increases towards Bayesian BFs. Thus we can relatively simply generalize the previous results.

Theorem 4. *Any symmetric quasi-Bayesian belief function $Bel_S = (s, s, ...s)$ is non-conflicting with any other belief function on a general frame of discernment Ω_n, i.e.. for any Bel and any symmetric BF Bel_S both on Ω_n it holds that $\mathbb{C}(Bel, Bel_S) = 0$. Specially, it holds $\mathbb{C}(Bel, U_n) = 0$.*

Proof. $AU(Bel_S) = AU(U_n) = log_2 n$, i.e. max possible uncertainty on Ω_n, $AU(Bel \oplus Bel_S)$ and $AU(Bel)$ lay on the same h-line thus $AU(Bel \oplus Bel_S) \leq AU(Bel)$, hence $\mathbb{C}(Bel, Bel_S) = 0$. For combinability see [17]. □.

Theorem 5. *(Categorical singletons) Let Bel_ω be a* categorical singleton*, i.e., belief function such that $m_\omega(\{\omega\}) = 1$ for some $\omega \in \Omega_n$ and $m_\omega(\{\omega'\}) = 0$ for $\omega \neq \omega' \in \Omega_n$ and Bel be any quasi-Bayesian BF on Ω_n combinable with Bel_ω. It hold that $\mathbb{C}(Bel, Bel_\omega) = 0$.*

Proof. It holds that $Bel_\omega \oplus Bel = Bel_\omega$ (in combinable case), thus $AU(Bel_\omega \oplus Bel) = AU(Bel_\omega) = 0$. Hence also $\mathbb{C}(Bel, Bel_\omega) = 0$. □.

Note that $\mathbb{C}(Bel, Bel_\omega)$ is not defined for Bel which core does not include ω because $Bel_\omega \oplus Bel$ is not defined there. From the same reason $\mathbb{C}(Bel_1, Bel_2)$ is not deifined either for any pair of BFs with disjunctive cores $(C)_1 \cap (C)_2 = \emptyset$. Hence full/total conflict is not defined by Harmanec degree of conflict $\mathbb{C}$.

Analysing situations analogous to those decribed for Ω_2, see Figures 7 — 8 we obtain:

Theorem 6. *(max C) Let Bel_u be a fixed quasi-Bayesian belief function on Ω_n and Bel any qBBF on Ω_n combinable with Bel. Maximal $\mathbb{C}(Bel, Bel_u)$ appears for a Bayesian BF Bel_m, which lies between BBF $-h(Bel_u)$ and border of $n-1$ dimensional simplex of BBFs in the directions opposite to the direction to BBF $h(Bel)$. $\mathbb{C}$ decreases from Bel_m in any direction.*

Proof. (to be typed)

Theorem 7. *Let us suppose two combinable quasi-Bayesian belief functions Bel_1 and Bel_2 on a general frame of discernment Ω_n. If Pl-$C(Bel_1, Bel_2) = 0$ then also $\mathbb{C}(Bel_1, Bel_2) = 0$.*

Proof. (to be typed)

Hypothesis 1 *Let us suppose two combinable quasi-Bayesian belief functions Bel_1 and Bel_2 on Ω_n. It holds that*

$$\mathbb{C}(Bel_1, Bel_2) \leq Pl\text{-}C(Bel_1, Bel_2).$$

6.3.2 A Comparison of the Approaches for General Belief Functions.

Because proof of Theorem 5 holds for any BF Bel we can simply formulate it also for general BFs.

Theorem 8. *(categorical singleton) Let Bel_ω be a* categorical singleton*, i.e., belief function such that $m_\omega(\{\omega\}) = 1$ for some $\omega \in \Omega_n$ and and $m_\omega(X) = 0$ for $\{\omega\} \neq X \subset \Omega_n$ and Bel be any BF on Ω_n combinable with Bel_ω. It hold that $\mathbb{C}(Bel, Bel_\omega) = 0$.*

Nevertheless, situation is much more complicated for general belief functions, as there are multi-dimensional structures instead of one-dimensional h-lines on Ω_n.

Due to this, we can observe a difference in common properties of conflicts between belief functions which are not quasi Bayesian. Thus a symmetric BF Bel_S (even U_n) is not non-conflicting with with any BF in general. Thus we have not a simple generalization of Theorems 7 and 3, because, e.g., there is always $Pl\text{-}C(Bel, Bel_S) = 0$, but there are situations for which $\mathbb{C}(Bel_1, Bel_2) > 0$ thus $\not\leq Pl\text{-}C(Bel, Bel_S)$. See following examples:

Example 1. Let $m_1(\{\omega_1\}) = \frac{1}{2}, m_1(\omega_2, \omega_3\}) = \frac{1}{2}$; $Bel_2 = (\frac{2}{10}, \frac{2}{10}, \frac{2}{10}, \frac{1}{10}, \frac{1}{10}, \frac{1}{10}; \frac{1}{10})$; $Bel_3 = (\frac{1}{6}, \frac{1}{6}, \frac{1}{6}, \frac{1}{6}, \frac{1}{6}, \frac{1}{6}; 0)$ $AU(Bel_1) = -\frac{1}{2}log_2\frac{1}{2} - 2\frac{1}{4}log_2\frac{1}{4} = 1.500$, $AU(Bel_2) = AU(Bel_3) = AU(U_3) = -3\frac{1}{3}log_2\frac{1}{3} = 1.585$. $Bel_1 \oplus Bel_2 = (\frac{4}{11}, \frac{3}{11}, \frac{3}{11}, 0, 0, \frac{1}{10}; 0)$, $AU(Bel_1 \oplus Bel_2) = -\frac{8}{22}log_2\frac{8}{22} - 2\frac{7}{22}log_2\frac{7}{22} = 1.582$; $Bel_1 \oplus Bel_3 = (\frac{3}{8}, \frac{2}{8}, \frac{2}{8}, 0, 0, \frac{1}{8}; 0)$, $AU(Bel_1 \oplus Bel_3) = -\frac{6}{16}log_2\frac{6}{16} - 2\frac{5}{16}log_2\frac{5}{16} = 1.579$.
Thus $\mathbb{C}(Bel_1, U_3) = AU(Bel_1 \oplus U_3) - AU(Bel_1) = AU(U_3) - AU(Bel_1) = 0.085 > 0, \mathbb{C}(Bel_1, Bel_2) = AU(Bel_1 \oplus Bel_2) - AU(Bel_1) = 0.082 > 0, \mathbb{C}(Bel_1, Bel_3) = AU(Bel_1 \oplus Bel_3) - AU(Bel_1) = 0.079 > 0$.
On the other hand we have $\mathbb{C}(Bel_1, VBF) AU(Bel_1 \oplus VBF) - AU(Bel_1) = AU(Bel_1) - AU(Bel_1) = 0$ as we expected; further for modified $Bel_1' = (\frac{1}{4}, 0, 0, 0, 0, \frac{1}{14}; \frac{1}{2})$ we obtain $AU(Bel_1') = AU(U_3)$, thus $\mathbb{C}(Bel_1', Bel_i) = max(0, AU(Bel_1' \oplus Bel_i) - AU(U_3) = 0$ in all 3 cases.

We have examples where U_3 and symmetric BFs are non-conflicting with other BFs and also counter-examples. Thus there arise an interesting open problem to specify conditions under which assertion of Theorem 7 holds for general BFs on general frame of discernment (as the cases where it does not hold are exeptions which should be specified), i.e., to specify under which contidions U_n, Bel_S and $Bel_{S_{Pl}}$ (and $Bel_{S_{Bet}}$) are non-conflicting with any others. (a generalization of Theorem 4). A special subproblem is specification under which conditions for Bel_S and $Bel_{S_{Pl}}$ holds that $AU(Bel_S \oplus Bel) \leq AU(Bel)$.

The related interesting open question is also generalization of Theorem 3 (including verification of Hypothesis 1), i.e, again a specification of conditions under which the Theorem is generalizable.

6 SUMMARY

We have seen that $\mathbb{C}(Bel_i, Bel_j)$ is a weaker measure of conflict than $Pl\text{-}C(Bel_i, Bel_j)$ on quasi-Bayesian BFs in the sense, that all non-conflicting couples of qBBFs with respect to $Pl\text{-}C$ are also non-conflicting with respect to $\mathbb{C}$. Moreover we have Hypothesis $\mathbb{C}(Bel_i, Bel_j) \leq Pl\text{-}C(Bel_i, Bel_j)$, which has already been proved on two-element frames of discernment . This is important as $Pl\text{-}C$ classifies as non-conflicting many cases which are considered to be positively conflicting by the other measures of conflict ($m(\emptyset)$, distances, Liu's cf, Martin's approach, Destercke-Burger's approach, ...).

On the other hand, there are several properties of Harmanec's degree if conflict $\mathbb{C}$ which seem surprising or even strange and which are significantly different even from plausibility conflict $Pl\text{-}C$: e.g., decreasing of conflict in the direction to categorical singletons ($m_\omega(\{\omega\}) = 1$) and non-conflictnes of categorical singletons with all combinable BFs, maximally conflicting BFs to given Bel_u located between $-h(Bel)$ and the border of the simplex of BBFs, non-conflicting areas according to conditions (iii) and (iv) from Theorem 1. This 'strange' behaviour is based on a completely different assumptions. Harmanec's conflict does not measure either difference or opposition of belief, but increasing/decreasing of uncertainty when BFs are combined, thus this 'strange' property of $\mathbb{C}$-conflict is sound from its point of view.

All of these properties should be discussed (accepted or explicitly rejected) when a general axiomatic approach to conflicts between belief functions will be formulated based on Destercke & Burger [12], Martin's [23] and author's approaches [6, 9, 11].

When using $\mathbb{C}$ we have to be carefull about values (specially about values around 1) as rounding of the values may produce relatively different results, see Example 3.

Example 3. Let us suppose $Bel_1 : \; m_1(\{\omega_1, \omega_2\}) = 0.999999, m_1(\Omega_5) = 0.000001$ and $Bel_2 : \; m_2(\{\omega_3\}) = 0.45, m_2(\{\omega_4\}) = 0.25, m_2(\{\omega_3, \omega_4, \omega_5\}) = 0.30$. Thus there is a high conflict $\mathbb{C}(Bel_1, Bel_2)$.
Let us round the inputs to 4 decimal places now. We obtain $Bel'_1 : \; m'_1(\{\omega_1, \omega_2\}) = 1.0000, m'_1(\Omega_5) = 0.0000$. $Bel'_2 = Bel_2$. We have $Pl\text{-}C(Bel'_1, Bel'_2) \doteq Pl\text{-}C(Bel_1, Bel_2)$ but a completely different $\mathbb{C}(Bel'_1, Bel'_2) = 0$ now!

A disadvantage of $\mathbb{C}$ is its strong relation to Dempster's rule of combination, thus $\mathbb{C}$ is applicable only in the classic Dempster-Shafer approach with the Dempster's rule.

7 CONCLUSION

Two completely different approaches to conflict of belief functions were analysed and compared. The common features were observed and the significant difference in behaviour was explained. The warning for application of Harmanec's conflict was presented.

The theoretic analysis and comparison of the approaches coming from significantly different assumptions move us to better understanding of nature of conflicts of belief functions in general. This can consequently serve as a basis for better combination of conflicting belief functions in future, whenever conflicting belief functions appear.

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References

1. Almond, R. G.: Graphical Belief Modeling. Chapman & Hall, London (1995)
2. Cobb, B. R., Shenoy, P. P.: A Comparison of Methods for Transforming Belief Functions Models to Probability Models. In: Nielsen, T.D., Zhang, N.L. (eds.) ECSQARU 2003. LNAI 2711, 255–66. Springer, Heidelberg (2003)
3. Daniel, M.: Algebraic structures related to Dempster-Shafer theory. In: Bouchon-Meunier, B., Yager, R. R., Zadeh, L. A. (eds.) *Advances in Intelligent Computing - IPMU'94*. LNCS 945, 51–61, Springer-Verlag, Berlin Heidelberg (1995)
4. Daniel, M.: Distribution of Contradictive Belief Masses in Combination of Belief Functions. In: B. Bouchon-Meunier, R. R. Yager, L. A. Zadeh (eds.) Information, Uncertainty and Fusion, pp. 431–446. Kluwer Academic Publishers, Boston (2000)
5. Daniel, M.: M. Daniel: Probabilistic Transformations of Belief Functions. In: Godo, L. (ed.) Symbolic and Quantitative Approaches to Reasoning with Uncertainty (ECSQARU 2005). LNAI 3571, 539–551. Springer, Heidelberg (2005)
6. Daniel, M.: Conflicts within and between Belief Functions. In: Hüllermeier, E., Kruse, R., Hoffmann, E. (eds.) *IPMU 2010*. LNAI 6178, 696–705, Springer-Verlag, Berlin Heidelberg (2010)
7. Daniel, M.: Non-conflicting and Conflicting Parts of Belief Functions. In: Coolen, F., de Cooman, G., Fetz, T., Oberguggenberger, M. (eds.) *ISIPTA'11; Proc. of the 7th ISIPTA*, pp. 149–158. Studia Universitätsverlag, Innsbruck (2011)
8. Daniel, M.: Introduction to an Algebra of Belief Functions on Three-element Frame of Discernment — A Quasi Bayesian Case. In: S. Greco et al. (eds.) *Advances in Computational Intelligence, IPMU 2012, Part III.* CCIS, vol. 299, pp 532–542. Springer, Berlin Heidelberg, 2012.
9. Daniel, M.: Properties of Plausibility Conflict of Belief Functions. In:Rutkowski, L., Korytkowski, M., Scherer, R., Tadeusiewicz, R., Zadeh, L.A., Zurada, J.M. (eds.) *ICAISC 2013, Part I.* LNCS, vol. 7894, pp. 235–246. Springer, Heidelberg (2013)
10. Daniel, M.: Belief Functions: a Revision of Plausibility Conflict and Pingnistic Conflict. In: W. Liu et al. (eds.) SUM 2013. LNAI 8078, 190–203, Springer-Verlag Berlin Heidelberg (2013)
11. Daniel, M.: Conflict between Belief Functions: a New Measure Based on their Non-Conflicting Parts. In: F. Cuzzolin (eds.) *BELIEF 2014*. LNCS (LNAI) vol. 8764, pp. 321–330. Springer, Heidelberg (2014)
12. S. Destercke and T. Burger. Revisiting the Notion of Conflicting Belief Functions. In: T. Denœux, M. H. Masson (eds.) *Belief Functions: Theory and Applications*, AISC 164, pp 153–160. Springer, Berlin Heidelberg (2012)
13. D. Dubois, H. Prade: Representation an combination of uncertainty with belief functions and possibility measures. *Computational Intelligence*, **4**: 244–264 (1988)
14. D. Dubois, H. Prade: Consonant Approximations of Belief Functions. *International Journal of Approximate Reasoning* **4**: 419–649 (1990)

15. Hájek, P., Havránek, T., Jiroušek, R.: *Uncertain Information Processing in Expert Systems.* CRC Press, Boca Raton, Florida (1992)
16. Hájek, P., Valdés, J. J.: Generalized algebraic foundations of uncertainty processing in rule-based expert systems (dempsteroids). *Computers and Artificial Intelligence* **10** (1), 29–42 (1991)
17. Harmanec, D.: A Note on Uncertainty, Dempster Rule of Combination, and Conflict. *Int. J. General Systems* **26** (1–2), 63–72 (1997)
18. Harmanec, D., Klir, G. J.: Measuring total uncertainty in Dempster-Shafer theory: A novel approach. *International Journal of General Systems* **22** (4), 405–419 (1994)
19. Harmanec, D., Resconi, G., Klir, G. J., Pan, Y.: On the computation of the uncertainty measure for the Dempster-Shafer theory. *Int. J. General Systems* **25** (2), 153–163 (1996)
20. Jousselme, A.-L., Grenier, D., Bossé, E.: A new distance between two bodies of evidence. *Information Fusion* **2** (2), 91–101 (2001)
21. E. Lefevre, Z. Elouedi, D. Mercier: Towards an alarm for opposition conflict in a conjunctive combination of belief functions. In: W. Liu (ed.), *ECSQARU 2011.* LNAI 6717: 314–325. Springer-Verlag, Berlin Heidelberg (2011)
22. Liu, W.: Analysing the degree of conflict among belief functions. Artificial Intelligence **170**, 909–924 (2006)
23. A. Martin. About Conflict in the Theory of Belief Functions. In: T. Denœux, M. H. Masson (eds.) *Belief Functions: Theory and Applications*, AISC 164, pp 161–168. Springer, Berlin Heidelberg, 2012.
24. Martin, A., Jousselme, A.-L., Osswald, C.: Conflict measure for the discounting operation on belief functions. In: *Fusion 2008*, Proceedings of 11th International Conference on Information Fusion. Cologne, Germany (2008)
25. Roquel, A., et al.: Decomposition of conflict as a distribution on hypotheses in the framework on belief functions. *Int. J. of Approx. Reasoning* **55**, 1129–1146 (2014)
26. Shafer, G.: *A Mathematical Theory of Evidence.* Princeton University Press, Princeton, New Jersey (1976)
27. Smets, Ph.: Analyzing the combination of conflicting belief functions. *Information Fusion* **8**, 387–412 (2007)
28. R. R. Yager: On the Demspter-Shafer framework and new combination rules. *Information Sciences*, **41**: 93–138 (1987)